

Bounding a global red-blue proportion using local conditions

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Abstract

We study the following local-to-global phenomenon: Let B and R be two finite sets of (blue and red) points in the Euclidean plane \mathbb{R}^2 . Suppose that in each “neighborhood” of a red point, the number of blue points is at least as large as the number of red points. We show that in this case the total number of blue points is at least one fifth of the total number of red points. We also show that this bound is optimal and we generalize the result to arbitrary dimension and arbitrary norm using results from Minkowski arrangements.

1 Introduction

Consider the following scenario in wireless networks. Suppose we have n clients and m antennas where both are represented as points in the plane (see Figure 1). Each client has a wireless device that can communicate with the antennas. Assume also

that each client is associated with some disk centered at the client’s location and having radius representing how far in the plane his device can communicate. Suppose also, that some communication protocol requires that in each of the clients disks, the number of antennas is at least some fixed proportion $\lambda > 0$ of the number of clients in the disk. Our question is: does such a local requirement imply a global lower bound on the number of antennas in terms of the number of clients? In this paper we answer this question and provide exact bounds. Let us formulate the problem more precisely.

Let B and $R = \{p_1, \dots, p_n\}$ be two finite sets in \mathbb{R}^2 . Let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a set of Euclidean disks centered at the red points, i.e., the center of D_i is p_i . Let $\{\rho_1, \dots, \rho_n\}$ be the radii of the disks in \mathcal{D} .

Theorem 1.1. *Assume that for each i we have $|D_i \cap B| \geq |D_i \cap R|$. Then $|B| \geq \frac{n}{5}$. Furthermore, the multiplicative constant $\frac{1}{5}$ cannot be improved.*

Such a local-to-global ratio phenomenon was shown to be useful in a more combinatorial setting. Pach et. al. [PRT15], solved a conjecture by Richter and Thomassen

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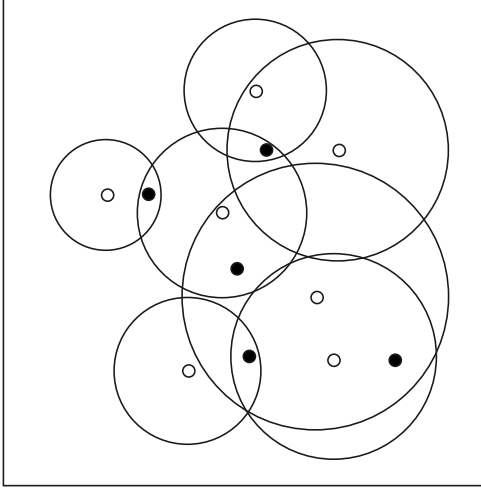


Figure 1: In each device range (each disk) there are at least as many antennas (black dots) as devices (white dots), so the hypothesis holds for $\lambda = 1$.

[RT95] on the number of total “crossings” that a family of pairwise intersecting curves in the plane in general position can have. Lemma 1 from their paper is a first step in the proof and it consists of a local-to-global phenomenon as described above.

We will obtain Theorem 1.1 from a more general result. In order to state it, we introduce some terminology.

Let K be an origin-symmetric convex body in \mathbb{R}^d , that is, the unit ball of a norm.

A *strict Minkowski arrangement* is a family $\mathcal{D} = \{K_1 = p_1 + \rho_1 K, \dots, K_n = p_n + \rho_n K\}$ of homothets of K , where $p_i \in \mathbb{R}^d$ and $\rho_i > 0$, such that no member of the family contains the center of another member. An *intersecting family* is a family of sets that all share at least one element.

We denote the *maximum cardinality of an intersecting strict Minkowski arrangement* of homothets of K by $M(K)$. It is known that $M(K)$ exists for every K and $M(K) \leq 3^d$ (see, e.g., Lemma 21 of [NPS16]). On the other hand (somewhat surprisingly), there is an origin-symmetric convex body K in \mathbb{R}^d such that $M(K) = \Omega(\sqrt{7}^d)$, [Tal98, NPS16]. For more on

Minkowski arrangements see, e.g., [FL94].

We need the following auxiliary Lemma.

Lemma 1.2. *Let K be an origin-symmetric convex body in \mathbb{R}^d . Let $R = \{p_1, \dots, p_n\}$ be a set of points in \mathbb{R}^d and let $\mathcal{D} = \{K_1 = p_1 + \rho_1 K, \dots, K_n = p_n + \rho_n K\}$ be a family of homothets of K . Then there exists a subfamily $\mathcal{D}' \subset \mathcal{D}$ that covers R and forms a strict Minkowski arrangement. Moreover, \mathcal{D}' can be found using a greedy algorithm.*

As a corollary, we will obtain the following theorem.

Theorem 1.3. *Let K be an origin-symmetric convex body in \mathbb{R}^d . Let $R = \{p_1, \dots, p_n\}$ be a set of points in \mathbb{R}^d and let $\mathcal{D} = \{K_1 = p_1 + \rho_1 K, \dots, K_n = p_n + \rho_n K\}$ be a family of homothets of K where $\rho_1, \dots, \rho_n > 0$. Let B be another set of points in \mathbb{R}^d , and assume that, for some $\lambda > 0$, we have*

$$\frac{|B \cap K_i|}{|R \cap K_i|} \geq \lambda, \quad (1)$$

for all $i \in [n]$. Then $\frac{|B|}{|R|} \geq \frac{\lambda}{3^d}$.

In Theorem 1.1 the convex body K is a Euclidean unit disk in the plane. Another case of special interest is when the convex body K is a unit cube and thus it induces the ℓ_∞ norm. In this situation we get a sharper and optimal inequality.

Theorem 1.4. *If K is the unit cube in \mathbb{R}^d , then the conclusion in Theorem 1.3 can be strengthened to $\frac{|B|}{|R|} \geq \frac{\lambda}{2^d}$. Furthermore, the multiplicative constant $\frac{1}{2^d}$ cannot be improved.*

In the results above, the points p_i play the role of the centers of the sets of the Minkowski arrangement. One might ask if this restriction is essential. As a final result, we give a general construction to show that it is.

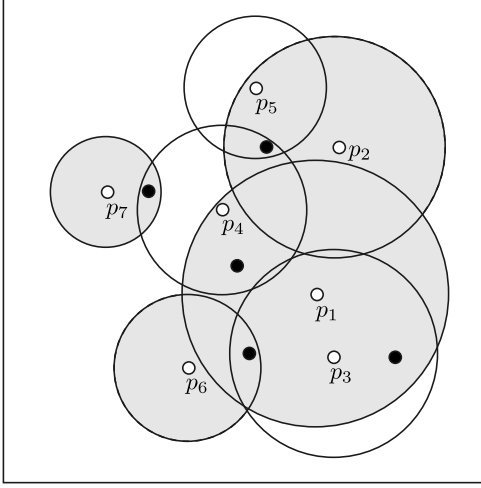


Figure 2: The centers of the disks are labeled in decreasing order of corresponding radii. The shaded disks cover the white points and no shaded disk contains the center of another.

Theorem 1.5. *Let K be any convex body in the plane and ε, λ any positive real numbers. There exist sets of points $R = \{p_1, \dots, p_n\}$ and B in the plane such that $|B| < \varepsilon n$ and that for each i there is a translate K_i of K that contains p_i for which $|B \cap K_i| \geq \lambda |R \cap K_i|$.*

In particular, even if each red point is contained in a unit disk with many blue points, the global blue to red ratio can be as small as desired. This is a possibly counter-intuitive fact in view of Theorem 1.1.

2 Proofs

Proof of Lemma 1.2. We construct a subfamily \mathcal{D}' of \mathcal{D} with the property that no member of \mathcal{D}' contains the center of any member of \mathcal{D}' , and $\bigcup \mathcal{D}'$ covers the red points, R . Assume without loss of generality that the labels of the points in R are sorted in non-increasing order of the homothety ratio, that is, $\rho_1 \geq \dots \geq \rho_n$. See Figure 2 for an example.

We construct \mathcal{D}' in a greedy manner as

follows: Add K_1 to \mathcal{D}' . Among all red points that are not already covered by \mathcal{D}' pick a point p_j whose corresponding homothet K_j has maximum homothety ratio ρ_j . Add K_j to \mathcal{D}' and repeat until all red points are covered by \mathcal{D}' . Note that the homothets in \mathcal{D}' are not necessarily disjoint.

Clearly, $R \subset \bigcup \mathcal{D}'$. Now we show that no member of \mathcal{D}' contains the center of another. Suppose to the contrary that K_i contains the center of K_j . If $i < j$, then $\rho_i \geq \rho_j$ so K_i was chosen first, a contradiction to the fact that p_j was chosen among the points not covered by previous homothets. If $i > j$, then K_j also contains the center of K_i , and we get a similar contradiction.

This finishes the proof of Lemma 1.2. \square

Proof of Theorem 1.3. By Lemma 1.2, there exists a subfamily $\mathcal{D}' \subset \mathcal{D}$ that covers R and form a strict Minkowski arrangement. Namely, $\bigcup \mathcal{D}'$ covers R , and no point of B is contained in more than $M(K)$ members of \mathcal{D}' . In particular, it follows that

$$|R| \leq \sum_{K \in \mathcal{D}'} |R \cap K| \leq \sum_{K \in \mathcal{D}'} \frac{|B \cap K|}{\lambda} \leq \frac{M(K)}{\lambda} |B|$$

so

$$\frac{|B|}{|R|} \geq \frac{\lambda}{M(K)} \geq \frac{\lambda}{3^d}.$$

This completes the proof. \square

Lemma 2.1. *Let K be the Euclidean unit disk centered at the origin. Then $M(K) = 5$.*

Proof of Lemma 2.1. Five unit disks centered in the vertices of a unit-radius regular pentagon show that $M(K) \geq 5$. See Figure 3a.

To prove the other direction, suppose that there is a point b in the plane that is contained in 6 Euclidean disks in a strict

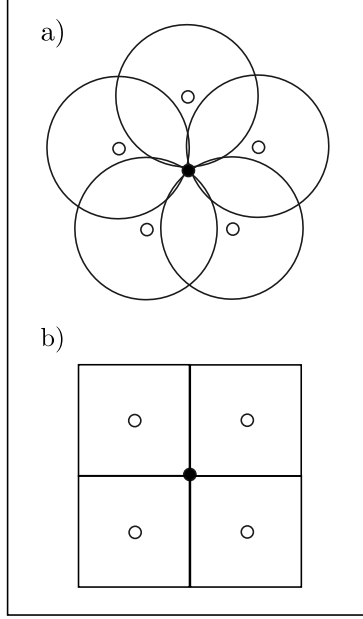


Figure 3: Optimal Minkowski arrangements in the plane for a) Euclidean disks, b) axis-parallel squares.

Minkowski arrangement. Then, by the pigeonhole principle, there are two centers of those disks, say p and q such that the angle $\angle(pbq)$ is at most 60° . Assume without loss of generality that $pb \geq qb$. It is easily verified e.g., by the law of cosines, that the distance pq is less than pb . Hence, the disk centered at p contains q , a contradiction. This completes the proof. \square

Lemma 2.2. *Let K be the unit cube of \mathbb{R}^d centered at the origin. Then $M(K) = 2^d$.*

Proof of Lemma 2.2. Let d be a positive integer and e_1, e_2, \dots, e_n the canonical base of \mathbb{R}^d . Consider all the cubes of radius 1 centered at each point of the form $\pm e_1 \pm e_2 \pm \dots \pm e_d$. This family shows that $M(K) \geq 2^d$. See Figure 3b for an example on the plane.

Now we show the other direction. Consider the 2^d closed regions of \mathbb{R}^d bounded by the hyperplanes $x_i = 0$ $i = 1, 2, \dots, d$ and suppose on the contrary that we have an example with $2^d + 1$ cubes or more that contain the origin. By the pigeon-hole principle there is a region with at least two cube

centers u and v . By applying a rotation we may assume that it is the region of vectors with non-negative entries. We may also assume $\delta := \|u\|_\infty \geq \|v\|_\infty$.

Since the d -cube centered at u contains the origin, its radius must be at least δ . We claim that this cube contains v . Indeed, each of the entries of u and v are in the interval $[0, \delta]$. So each of the entries of $u - v$ are in $[-\delta, \delta]$. Then $\|u - v\|_\infty \leq \delta$ as claimed. This contradiction finishes the proof. \square

Theorem 1.1 clearly follows from combining the proof of Theorem 1.3 (with $\lambda = 1$) and Lemma 2.1. The result is sharp because we have equality when R is the set of vertices of a regular pentagon with center p and $B = \{p\}$. Similarly, Theorem 1.4 and its optimality follow from Lemma 2.2.

Remark 2.3. *Lemma 2.1 can be generalized to arbitrary dimension. This implies that Theorem 1.1 can be generalized to arbitrary dimension almost verbatim.*

Proof of Theorem 1.5. Let K be any convex body in the plane. We construct sets R and B as follows. Let ℓ be a tangent line of K which intersects K at exactly one point t . Let I be a non-degenerate closed line segment contained in K and parallel to ℓ . Let J be the (closed) segment that is the locus of the point t as K varies through all its translations in direction d that contain I . See Figure 4.

We construct R by taking any n points from J and we construct B by taking any m points from I . For any point in R there is a translation of K that contains exactly one point of R and m points of B , which makes the local B to R ratio equal to m . But globally we can make the ratio $\frac{m}{n}$ arbitrarily small. \square

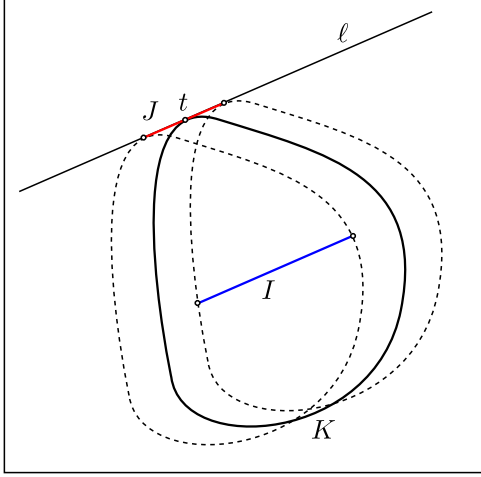


Figure 4: Construction of example without local-to-global phenomenon.

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